EE 508
Lecture 12

The Approximation Problem

Classical Approximating Functions

- Thomson and Bessel Approximations

Review from Last Time

Elliptic Filters

Magnitude-Squared Elliptic Approximating Function

$$H_E(\omega)=\frac{1}{1+\varepsilon^2 C_{Rn}^2(\omega)}$$

Inverse mapping to $T_E(s)$ exists

- For n even, n zeros on imaginary axis
- For n odd, n-1 zeros on imaginary axis

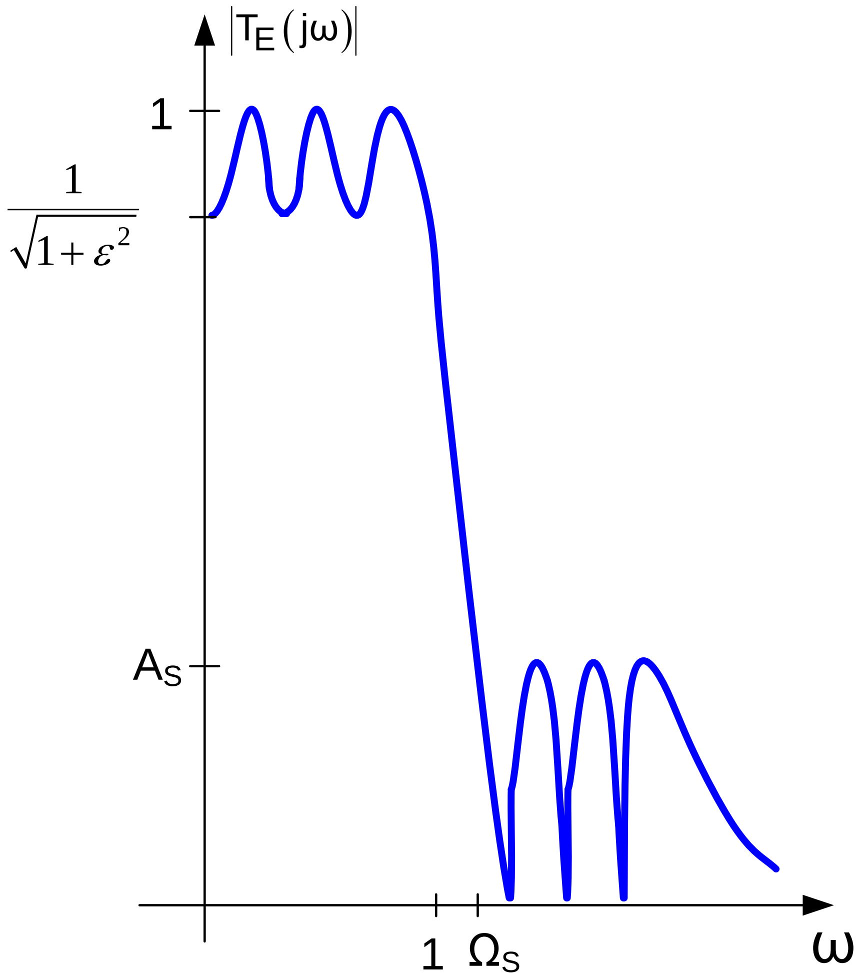
} Termed here "full order"

- Equal ripple in both pass band and stop band
- Analytical expression for poles and zeros not available
- Often choose to have less than n or n-1 zeros on imaginary axis

(No longer based upon CC rational fractions)

Review from Last Time

Elliptic Filters



- If of full-order, response completely characterized by $\{n,\varepsilon,A_S.\Omega_S\}$
- Any 3 of these paramaters are independent
- Typically ε,Ω_S, and A_S are fixed by specifications (i.e. must determine n)

Thomson and Bessel Approximations

- All-pole filters
- Maximally linear phase at $\omega=0$

Review from Last Time

Thomson and Bessel Approximations

Consider T(jω)

$$T(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_R(\omega)+jN_{IM}(\omega)}{D_R(\omega)+jD_{IM}(\omega)}$$

$$\text{phase} = \angle(T(j\omega)) = \tan^{-1}\left(\frac{N_{IM}(\omega)}{N_R(\omega)}\right) - \tan^{-1}\left(\frac{D_{IM}(\omega)}{D_R(\omega)}\right)$$

- Phase expressions are difficult to work with
- Will first consider group delay and frequency distortion

Review from Last Time

Linear Phase

Consider T(jω)

$$T(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{N_R(\omega)+jN_{IM}(\omega)}{D_R(\omega)+jD_{IM}(\omega)}$$

$$\angle\left(T(j\omega)\right) = \tan^{-1}\left(\frac{N_{IM}(\omega)}{N_R(\omega)}\right) - \tan^{-1}\left(\frac{D_{IM}(\omega)}{D_R(\omega)}\right)$$

Defn: A filter is said to have linear phase if the phase is given by the expression

$\angle\left(T(j\omega)\right) = \theta\omega$ where θ is a constant that is independent of ω

Preserving the waveshape

A filter has no frequency distortion for a given input if the output wave shape is preserved (i.e. the output wave shape is a magnitude scaled and possibly time-shifted version of the input)

Mathematically, no frequency distortion for $V_{IN}(t)$ if

$$V_{OUT}(t)=KV_{IN}(t-t_{shift})$$

Could have frequency distortion for other inputs

Review from Last Time

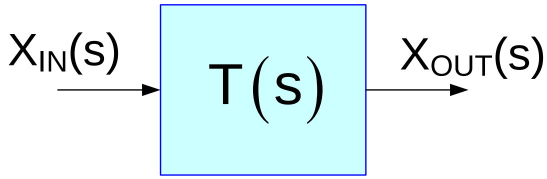
Preserving wave-shape in pass band

A filter is said to have linear passband phase if the phase in the passband of the filter is given by the expression $\angle\left(T\left(j\omega\right)\right) = \theta\omega$ where θ is a constant that is independent of ω

If a filter has linear passband phase in a flat passband, then the waveshape is preserved provided all spectral components of the input are in the passband and the output can be expressed as an amplitude scaled and time shifted version of the input by the expression

$$V_{OUT}(t) = KV_{IN}(t - t_{shift})$$

Amplitude (Magnitude) Distortion, Phase Distortion and Preserving wave-shape in pass band



Review from Last Time

Amplitude and phase distortion are often of concern in filter applications requiring a flat passband and a flat zero-magnitude stop band

Amplitude distortion is usually of little concern in the stopband of a filter

Phase distortion is usually of little concern in the stopband of a filter

A filter with no amplitude distortion or phase distortion in the passband and a zero-magnitude stop band will exhibit waveform distortion for any input that has a frequency component in the passband and another frequency component in the stopband

It can be shown that the only way to avoid magnitude and phase distortion respectively for signals that have energy components in the interval $\omega_1<\omega<\omega_2$ is to have constants k_1 and k_2 such that

$$\left.\begin{aligned} |T(j\omega)| &= k_1 \\ \angle T(j\omega) &= k_2\omega \end{aligned}\right\} \quad \text{for } \omega_1 < \omega < \omega_2$$

Group Delay

Defn: Group Delay is the negative of the phase derivative with respect to ω

$$\tau_G = -\frac{d\angle T(j\omega)}{d\omega}$$

Recall, by definition, the phase is linear iff $\angle T(j\omega) = k\omega$

If the phase is linear, $\tau_G = -\frac{d\angle T(j\omega)}{d\omega} = -\frac{d(k\omega)}{d\omega} = -k$

Thus for $\angle T(j0) = 0$, the phase is linear iff the group delay is constant

The group delay and the phase of a transfer function carry the same information

But, of what use is the group delay?

Group Delay

Example: Consider what is one of the simplest transfer functions

$$T(s)=\frac{1}{s+1}$$

$$T(j\omega)=\frac{1}{j\omega+1} \qquad \angle T(j\omega)= -\tan^{-1}\left(\frac{\omega}{1}\right) \qquad \tau_G=-\frac{d\angle T(j\omega)}{d\omega}$$

The phase of T(s) is analytically very complicated

$$\tau_G=-\frac{d\angle T(j\omega)}{d\omega}=-\frac{d\left(-\tan^{-1}\omega\right)}{d\omega}$$

Recall the identity

$$\frac{d\left(\tan^{-1}u\right)}{dx}=\left(\frac{1}{1+u^2}\right)\frac{du}{dx}$$

$$\tau_G=-\frac{d\left(-\tan^{-1}\omega\right)}{d\omega}=--\frac{1}{1+\omega^2}$$

Thus

$$\tau_G=\frac{1}{1+\omega^2}$$

Note that the group delay is a rational fraction in ω^2 instead of an arctan function

Group Delay

But, of what use is the group delay?

The phase of almost all useful transfer functions are complicated functions involving sums of arctan functions and these are difficult to work with analytically

Theorem: The group delay of any transfer function is a rational fraction in ω^2

From this theorem, it can be observed that the group delay is much more suited for analytical investigations than is the phase

Proof of Theorem:

(for notational convenience, will consider only all-pole transfer functions)

$$T(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem:

$$T(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

$$T(j\omega) = \frac{1}{\left(1 - a_2\omega^2 + a_4\omega^4 + \ldots\right) + j\omega\left(a_1 - a_3\omega^2 + a_5\omega^4 + \ldots\right)}$$

$$T(j\omega) = \frac{1}{F_1\left(\omega^2\right) + j\omega F_2\left(\omega^2\right)}$$

where F_1 and F_2 are even polynomials in ω

$$\angle T(j\omega) = -\tan^{-1}\left(\frac{\omega F_2\left(\omega^2\right)}{F_1\left(\omega^2\right)}\right)$$

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem: $$\angle T(j\omega) = -\tan^{-1}\left(\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right)$$

but from identity $$\frac{d\left(\tan^{-1} u\right)}{dx} = \left(\frac{1}{1+u^2}\right)\frac{du}{dx}$$

$$\tau_G = -\frac{d\angle T(j\omega)}{d\omega} = -\frac{1}{1+\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]^2} \bullet \frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega}$$

Now consider the right-most term in the product

$$\frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega} = \frac{F_1(\omega^2)\left[\frac{d\left(\omega F_2(\omega^2)\right)}{d\omega}\right] - \left(\omega F_2(\omega^2)\right)\frac{d\left(F_1(\omega^2)\right)}{d\omega}}{\left[F_1(\omega^2)\right]^2}$$

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem:

$$\angle T(j\omega) = -\tan^{-1}\left(\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right)$$

$$\frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega} = \frac{F_1(\omega^2)\left[\frac{d\left(\omega F_2(\omega^2)\right)}{d\omega}\right] - \left(\omega F_2(\omega^2)\right)\frac{d\left(F_1(\omega^2)\right)}{d\omega}}{\left[F_1(\omega^2)\right]^2}$$

Odd

Even

Even

Thus this term is an even rational fraction in ω

Group Delay

Theorem: The group delay of any transfer function is a rational fraction in ω^2

Proof of Theorem:

$$\tau_G = -\frac{d\angle T(j\omega)}{d\omega} = -\frac{1}{1+\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]^2} \bullet \frac{d\left[\frac{\omega F_2(\omega^2)}{F_1(\omega^2)}\right]}{d\omega}$$

Even

It follows that τ_G is the product of rational fractions in ω^2 so it is also a rational fraction in ω^2

Although tedious, the results can be extended when there are zeros present in T(s) as well

Thomson and Bessel Approximations

- All-pole filters
- Maximally linear phase at ω=0
- $$\left.\frac{d\angle T(j\omega)}{d\omega}\right|_{\omega=0} = -1$$

since $$\tau_G = -\frac{d\angle T(j\omega)}{d\omega}$$

These criteria can be equivalently expressed as

- All-pole filters
- Maximally constant group delay at ω=0
- $\tau_G = 1$ at ω=0

Thomson and Bessel Approximations

$$T_A(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

Must find the coefficients a_0, a_1,… an to satisfy the constraints

$$T(j\omega) = \frac{1}{\left(1 - a_2\omega^2 + a_4\omega^4 + \ldots\right) + j\omega\left(a_1 - a_3\omega^2 + a_5\omega^4 + \ldots\right)}$$

Theorem: If $T(j\omega) = \frac{1}{x + jy}$ then τ_G is given by the expression

$$\tau_G = \frac{x\frac{dy}{d\omega} - y\frac{dx}{d\omega}}{x^2 + y^2}$$

This theorem is easy to prove using the identity given above, proof will not be given here

Thomson and Bessel Approximations

$$T_A(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

Must find the coefficients a_0, a_1,… an to satisfy the constraints

$$T(j\omega) = \frac{1}{\left(1-a_2\omega^2+a_4\omega^4+...\right)+j\omega\left(a_1-a_3\omega^2+a_5\omega^4+...\right)}$$

From this theorem, it follows that

$$\tau_G = \frac{a_1+\omega^2\left(a_1a_2-3a_3\right)+\omega^4\left(5a_5-3a_1a_4+a_2a_3\right)+...}{1+\omega^2\left(a_1^2-2a_2\right)+\omega^4\left(a_2^2-2a_1a_3+2a_4\right)+...}$$

from the constraint $\tau_G = 1$ at ω=0, it follows that a_1=1

To make τ_G maximally constant at ω=0, want to match as many coefficients in the numerator and denominator as possible starting with the lowest powers of ω^2

from ω^2 terms $\quad a_1a_2-3a_3 = a_1^2-2a_2$

from ω^4 terms $\quad 5a_5-3a_1a_4+a_2a_3 = a_2^2-2a_1a_3+2a_4$

….

Thomson and Bessel Approximations

$$T_A(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

Must find the coefficients a_0, a_1,… a_n to satisfy the constraints

It can be shown that the a_ks are given by

$$a_k = \frac{(2n-k)!}{H2^{n-k}k!(n-k)!} \qquad \text{for } 1 \le k \le n-1$$

$$a_n = H$$

where

$$H = \frac{(2n)!}{2^n\ n!}$$

Thomson and Bessel Approximations

$$T_A(s) = \frac{1}{\sum_{k=0}^{n} a_k s^k}$$

Must find the coefficients a_0, a_1,… an to satisfy the constraints

Alternatively, if we define the recursive polynomial set by

$$B_1 = s+1$$

$$B_2 = s^2 + 3s + 3$$

$$\ldots$$

$$B_k = (2k-1)B_{k-1} + s^2 B_{k-2}$$

Then the n-th order Thompson approximation is given by

$$T_{An}(s) = \frac{B_n(0)}{B_n(s)}$$

Since the recursive set of polynomials are termed Bessel functions, this is often termed the Bessel approximation

Thomson and Bessel Approximations

Friedrich Bessel 1784-1846

Astronomer, Physicist, Mathematician

Was Bessel before his time in the filter field?

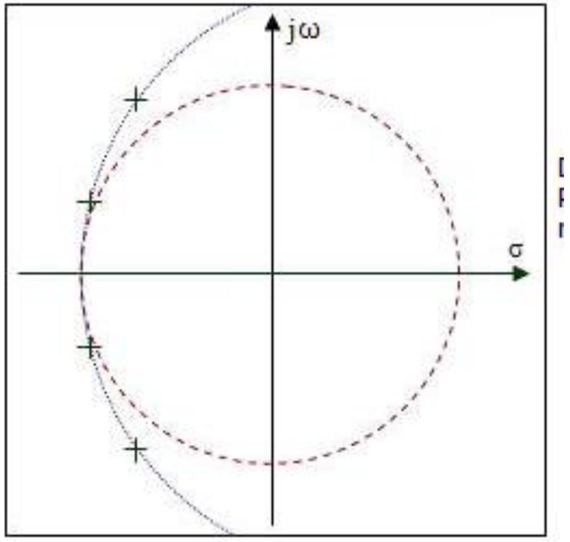
W.E. Thomson 1949

Z. Kiyasu 1943

} Applied to filter field

Thomson and Bessel Approximations

$$T_{An}(s) = \frac{B_n(0)}{B_n(s)}$$



http://www.rfcafe.com/references/electrical/bessel-poles.htm

- Poles of Bessel Filters lie on circle
- Circle does not go through the origin
- Poles not uniformly space on circumference

Thomson and Bessel Approximations

$$T_{An}(s) = \frac{B_n(0)}{B_n(s)}$$

Observations:

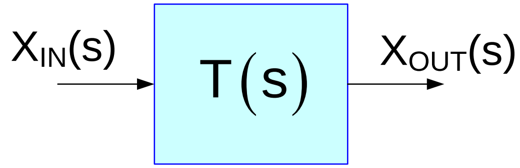
The Thomson approximation has relatively poor magnitude characteristic (at least if considered as an approximation to the standard lowpass function)

The normalized Thomson approximation has a group delay of 1 or a phase of ω at ω=0

Frequency scaling is used to denormalize the group delay or the phase to other values

Thomson and Bessel Approximations

Use of Bessel Filters:



Consider: $T(s) = e^{-sh}$ (not realizable but can be approximated)

$$T(j\omega) = e^{-j\omega h}$$

$$T(j\omega) = \cos(-\omega h) + j\sin(-\omega h)$$

$$|T(j\omega)| = 1 \qquad \angle T(j\omega) = -h\omega$$

If $x_{IN}(t) = X_M \sin(\omega t + \theta)$

$$x_{OUT}(t) = X_M \sin(\omega t + \theta - h\omega)$$

$$x_{OUT}(t) = X_M \sin(\omega[t-h] + \theta)$$

This is simply a delayed version of the input

$$x_{OUT}(t) = x_{IN}(t-h)$$

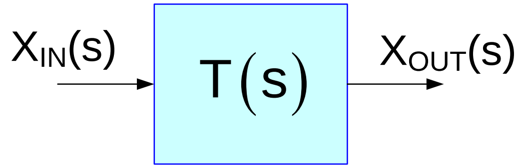
But

$$\tau_G = \frac{-d\angle T(j\omega)}{d\omega} = h \qquad x_{OUT}(t) = x_{IN}(t-\tau_G)$$

So, output is delayed version of input and the delay is the group delay

Thomson and Bessel Approximations

Use of Bessel Filters:



$$T(s) = e^{-sh}$$

$$\left|T(j\omega)\right| = 1 \qquad \angle T(j\omega) = -h\omega \qquad \tau_G = h$$

It is challenging to build filters with a constant delay

A filter with a constant group delay and unity magnitude introduces a constant delay

Bessel filters are filters that are used to approximate a constant delay

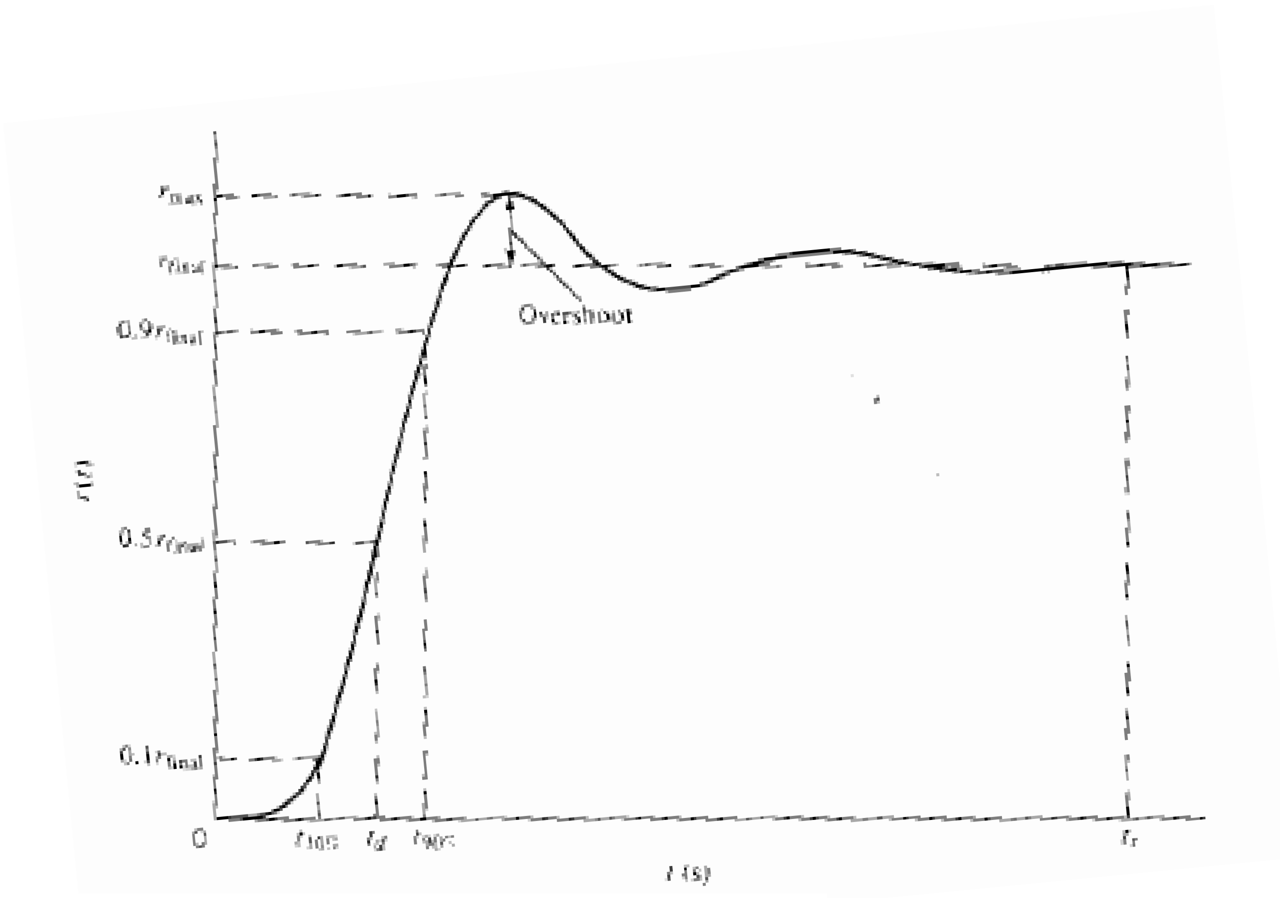
Bessel filters are attractive for introducing constant delays in digital systems

Some authors refer to Bessel filters as “Delay Filters”

An ideal delay filter would

- introduce a time-domain shift of a step input by the group delay
- introduce a time-domain shift of each spectral component by the group delay
- introduce a time-domain shift of a square wave by the group delay

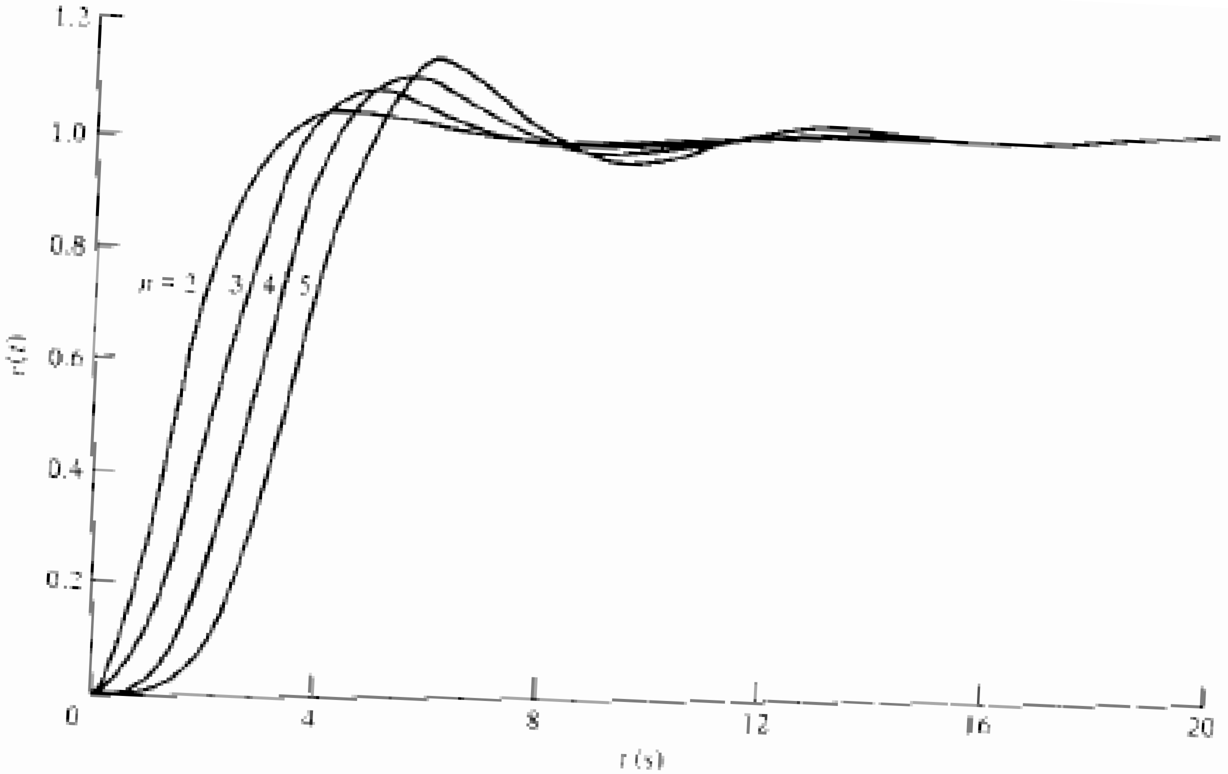
Thomson and Bessel Approximations



Characterization of the step response of a filter

From Introduction to the Theory and Design of Active Filters by Huelsman and Allen, p. 94-96

Thomson and Bessel Approximations



Step Response of Butterworth Filter

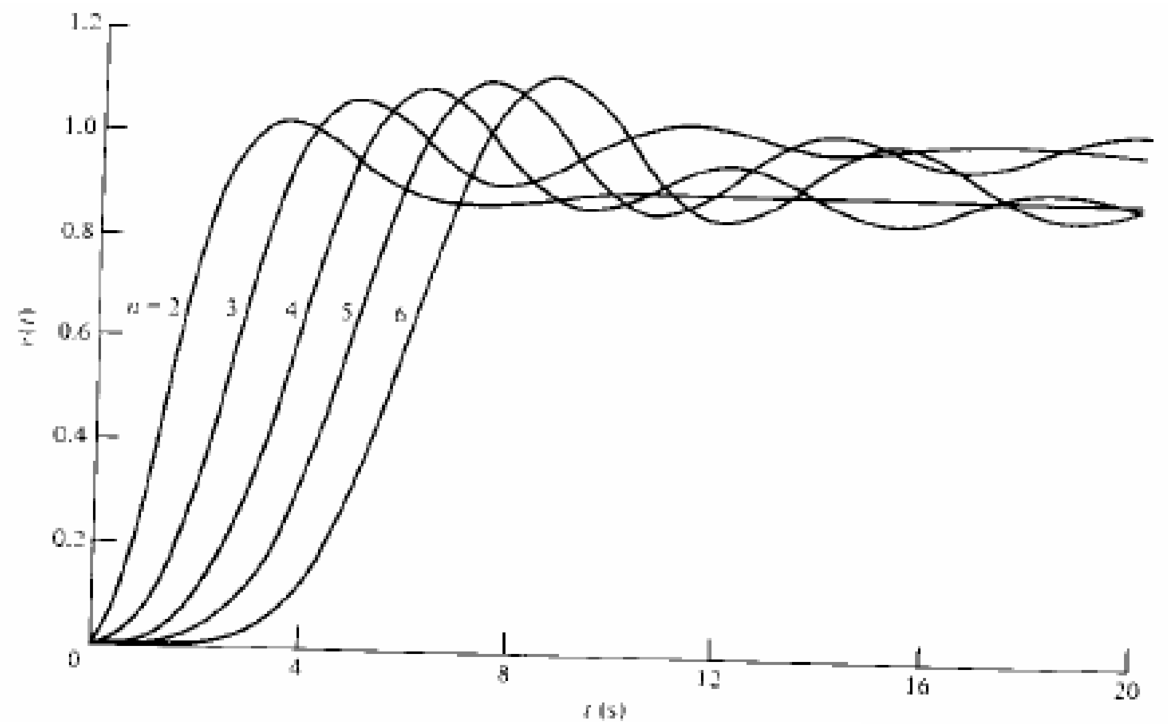
Delay is not constant

Overshoot present and increases with order

BW filters do not perform well as delay filters

From Introduction to the Theory and Design of Active Filters by Huelsman and Allen, p. 94-96

Thomson and Bessel Approximations

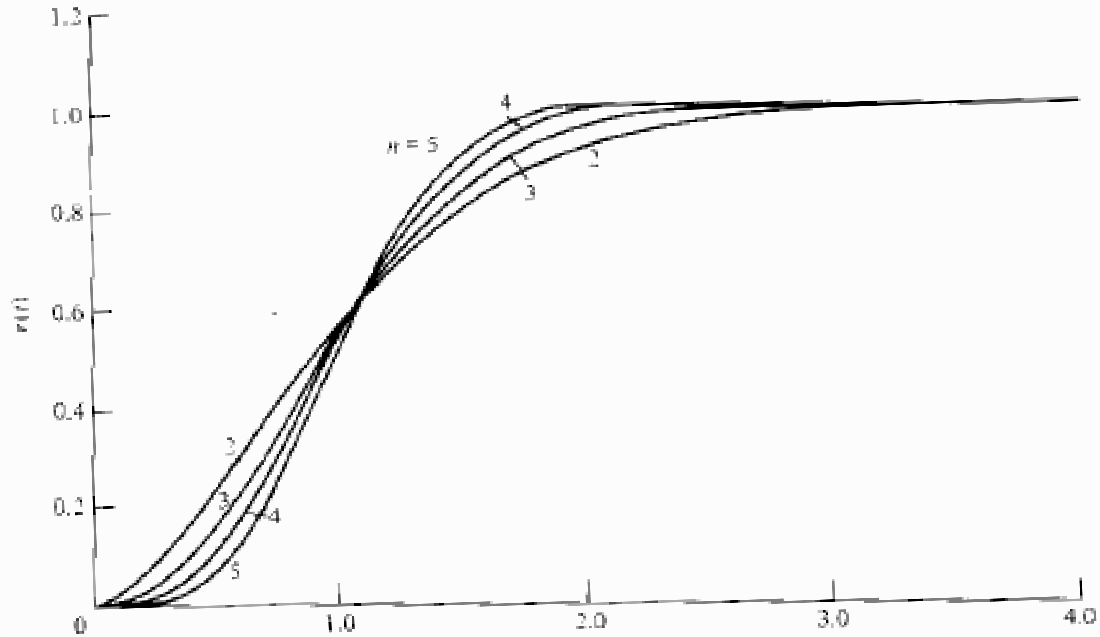


Step Response of Chebyschev Filter

Delay is not constant
Overshoot and ringing present and increases with order
CC filters do not perform well as delay filters

From Introduction to the Theory and Design of Active Filters by Huelsman and Allen, p. 94-96

Thomson and Bessel Approximations

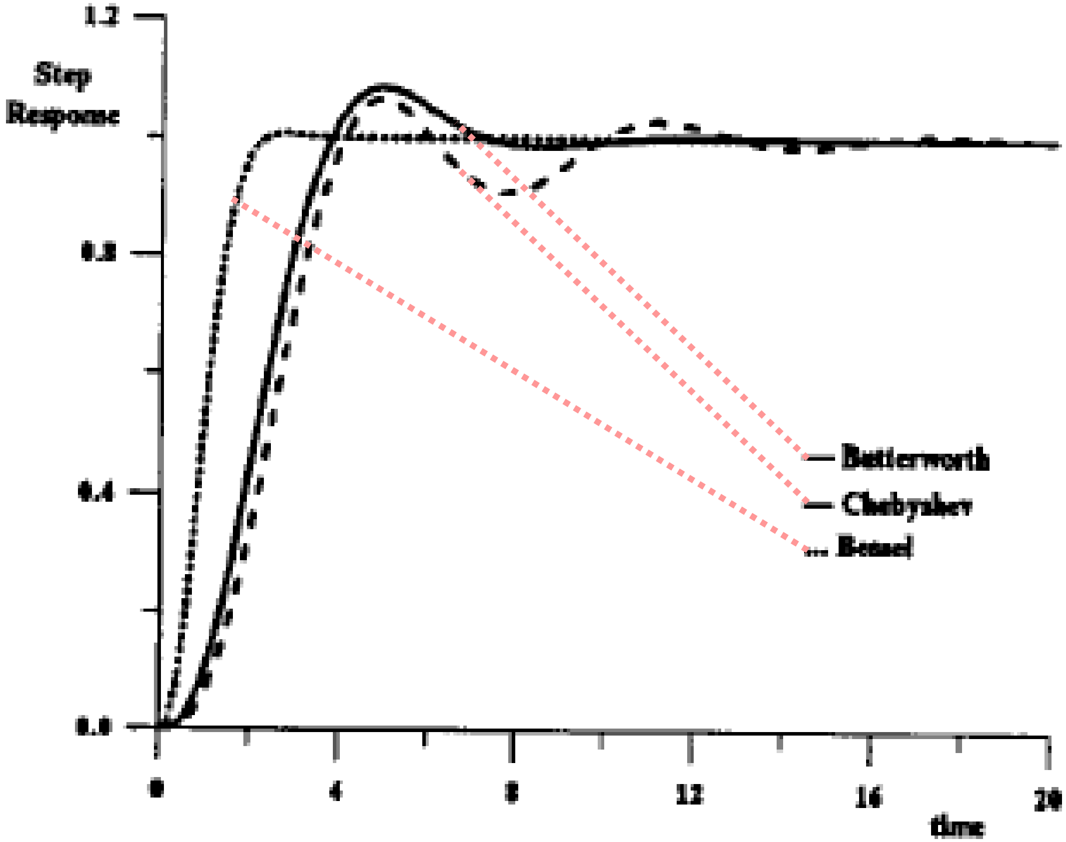


Step Response of Bessel Filters

Delay becomes more constant as order increases
No overshoot or ringing present
Bessel filters widely used as delay filters
Bessel filters often designed to achieve time-domain performance

From Introduction to the Theory and Design of Active Filters by Huelsman and Allen, p. 94-96

Thomson and Bessel Approximations



- Comparison of Step Response of 3rd-order Bessel, BW and CC filters
- Comparison for normalized frequency response for BW, CC and normalized group delay for Bessel

From Continuous-Time Active Filter Design by Deliyannis, Sun and Fidler

Thomson and Bessel Approximations

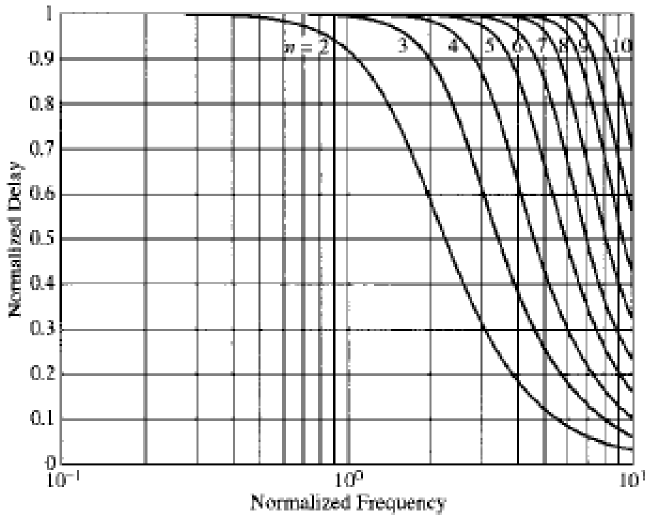


Figure 10.3 Delay of Bessel–Thomson filters of orders 2 through 10.

Harmonics in passband of Bessel Filter increase with n

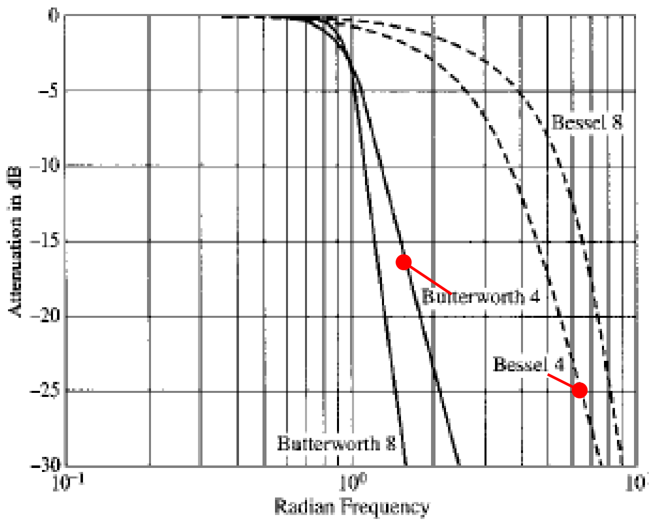
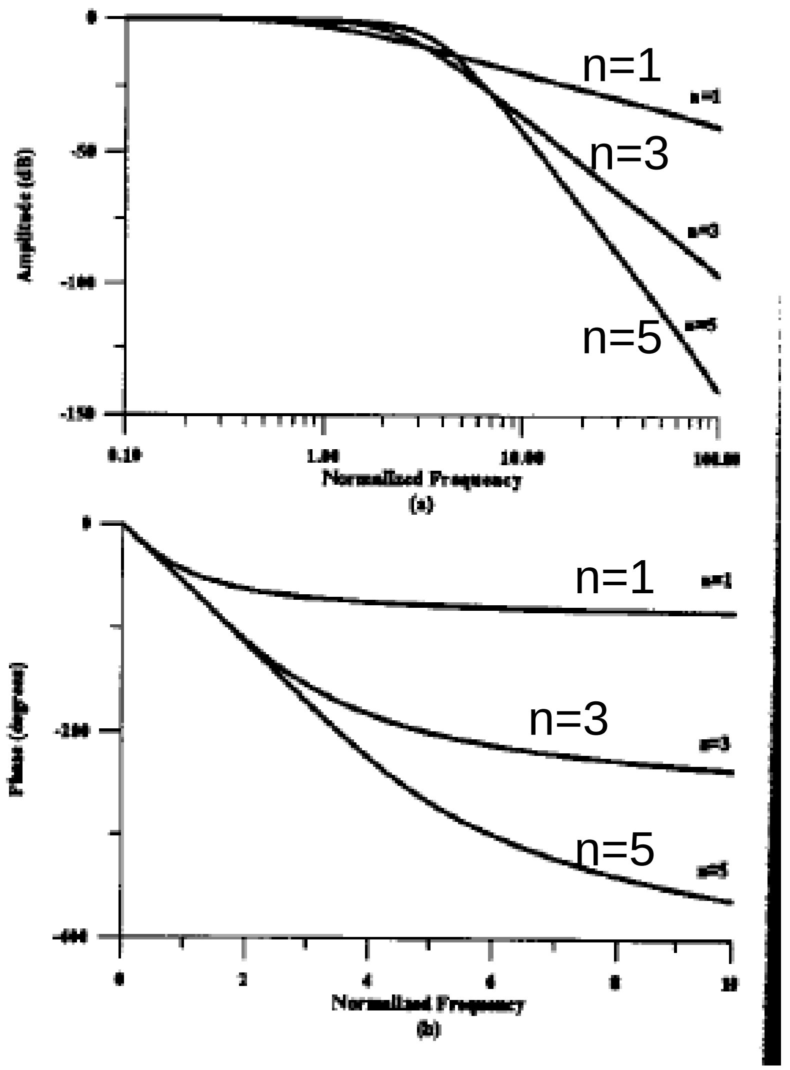


Figure 10.4 Comparison of Bessel–Thomson and Butterworth responses of orders 4 and 8.

Attenuation of amplitude for Bessel does not compare favorably wth BW, CC, or Eliptic filters

From Design of Analog Filters by Schaumann and Van Valkenburg, p405

Thomson and Bessel Approximations

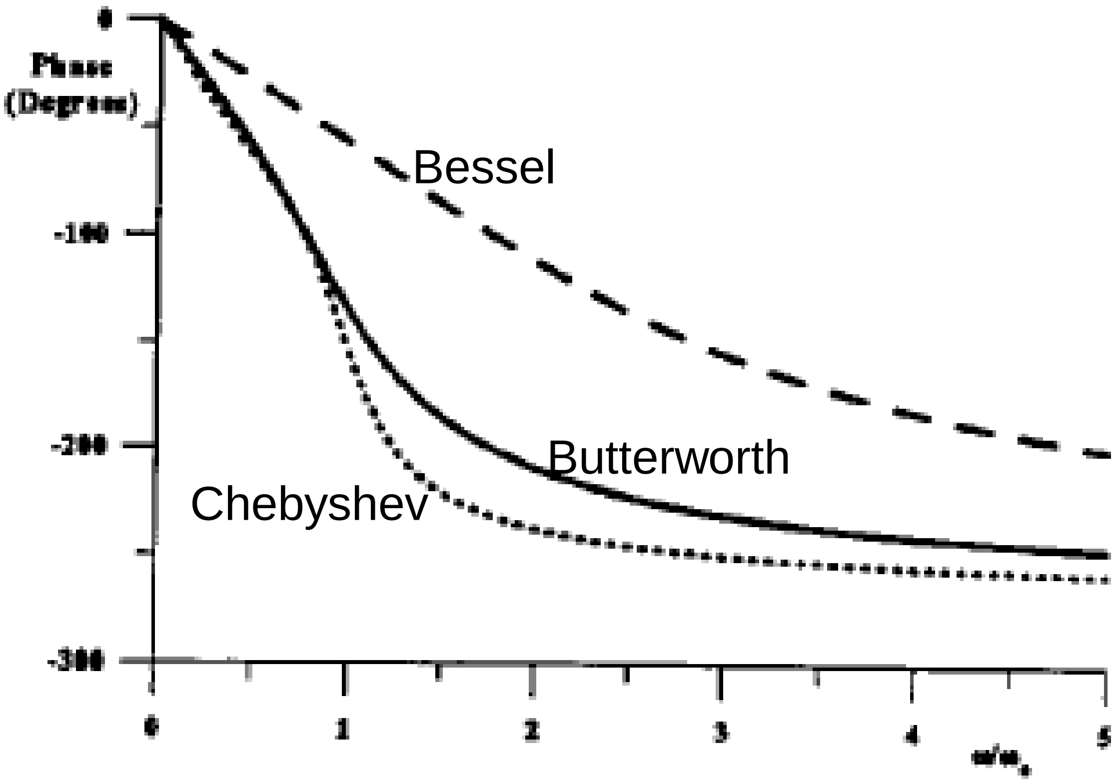


Magnitude of Bessel filters does not drop rapidly at band edge

Phase of Bessel filters becomes very linear in passband as order increases

From Continuous-Time Active Filter Design by Deliyannis, Sun and Fidler

Thomson and Bessel Approximations



Comparison of Phase Response of 3rd-order Bessel, BW and CC filters

From Continuous-Time Active Filter Design by Deliyannis, Sun and Fidler

Stay Safe and Stay Healthy !

End of Lecture 12